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1990 J. Phys. A: Math. Gen. 23 4417

(http://iopscience.iop.org/0305-4470/23/20/004)

J. Phys. A: Math. Gen. **23** (1990) 4417–4425. Printed in the UK

On the generalization of Molien functions to supergroups

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Received 12 February 1990, in final form 13 June 1990

Abstract. The construction of generating functions for the multiplicities of irreducible representation Γ in the decomposition of sth supersymmetrical power $\gamma\{s\}$, where Γ and γ are the representations of some supergroup G, is discussed. A number of examples are considered. The definitions are given for general groups, but the derived results are for the simplest specific cases.

1. Introduction

Recently superalgebras and supergroups have been widely applied to elementary particle theory (Fayet 1986), and in nuclear theory as well (Balantekin and Bars 1981). Therefore it is of current interest to generalize the various results of the representation theory of ordinary Lie groups and Lie algebras with a view to applying them to supergroups and superalgebras. A great deal has been done in this field by Berezin (1983) and other authors (Bernstein and Leits 1980, Leits 1980, Hughes and King 1987). In this paper we shall consider the generalization of the so-called Molien functions (Molien 1898) to supergroups. Molien functions play an important role in the theory of invariants and their generalization to the case of supergroups is very important for the construction of the effective Hamiltonians of the quantum system possessing one or another type of supersymmetry.

First we shall briefly recall the definition of Molien functions (Asherova *et al* 1988). Let us denote by γ a given (irreducible or reducible) representation of the group G. We shall denote by $\gamma^{[s]}$ the symmetrized sth Kronecker product

$$\gamma^{[s]} = \underbrace{\gamma \times \gamma \times \ldots \times \gamma}_{s\ \text{times}}$$

of this representation γ. The representation $\gamma^{[s]}$ is normally reducible and is decomposed into a direct sum of the irreducible representations Γ of G. This means that

$$\gamma^{[s]} = \sum_{\Gamma} n(\Gamma, \gamma, s)\Gamma \tag{1.1}$$

where $n(\Gamma, \gamma, s)$ is the multiplicity of Γ, i.e. the number of times it occurs in the decomposition of $\gamma^{[s]}$ into the direct sum of Γ. The Molien function (MF) $\Phi(\Gamma, \gamma; \lambda)$ is the generating function for $n(\Gamma, \gamma, s)$. That means that the expansion of Φ in powers of some parameter λ has the form

$$\Phi(\Gamma, \gamma; \lambda) = \sum_{s=0}^{\infty} n(\Gamma, \gamma, s)\lambda^{s}. \tag{1.2}$$

We shall call this function the boson MF.

0305-4470/90/204417+09\$03.50

Now (by considering a number of examples), it will be shown the way this function can be generalized to supergroups. We begin with the construction of Molien functions $\varphi(\Gamma, \gamma; \lambda)$ for the antisymmetrized product $\gamma^{[1^s]}$ of the representation γ. This function will be referred to as a fermion MF. The consideration of such an MF allows us to construct the MF for the graduated representation γ (supersymmetric Molien functions).

2. Fermion Molien functions

As has been mentioned above, the fermion MF $\varphi(\Gamma, \gamma; \lambda)$ is the generating function (GF) for the multiplicity $n(\Gamma, \gamma, s)$ of Γ in the decomposition of the antisymmetrized Kronecker product

$$\gamma^{[1^s]} = A\gamma \times \gamma \times \ldots \times \gamma$$

of the (reducible or irreducible) representation γ of the group G into the direct sum of irreducible representations Γ of G:

$$\gamma^{[1^s]} = \sum_{\Gamma} n(\Gamma, \gamma, s)\Gamma. \tag{2.1}$$

In other words, the expansion of $\varphi(\Gamma, \gamma, \lambda)$ in powers of λ has the form

$$\varphi(\Gamma, \gamma; \lambda) = \sum_{s=0}^{d} n(\Gamma, \gamma, s)\lambda^{s}. \tag{2.2}$$

(In contrast to (1.2) in this case the sum is finite and the number of terms does not exceed the dimension of γ, $d = \dim \gamma$.)

We shall consider in more detail the case when $G = SO(3)$ and when $\gamma = D^j$ is the irreducible representation of the three-dimensional rotation group with the highest weight j. Obviously $n(\Gamma, \gamma, s)$ can be calculated using the formula

$$n(\Gamma, \gamma, s) = \frac{1}{V_G} \int dg\, \chi_{\gamma^{[1^s]}}(g)\chi_{\Gamma}^{*}(g) \tag{2.3}$$

where $\chi_\Gamma(g) = \operatorname{Tr} D^\Gamma(g)$ is the character of the IR Γ of G and $V_G = \int dg$ is the volume of the group. The characters of the rotations through an angle ω for the arbitrary IR D of the group SO(3) are well known

$$\chi_J(\omega) = (z^{J+1} - z^{-J})/(z-1) \qquad z = \exp(i\omega). \tag{2.4}$$

Therefore it is necessary to calculate the character of the antisymmetrized sth Kronecker product $j^{[1^s]}$ of the representation D^j of the group SO(3), whose basis vectors are of the type $|m\rangle$, $-j \leqslant m \leqslant j$. The basis for the representation $j^{[1^s]}$ is formed by means of an antisymmetrized function of the type (Slater determinants)

$$|m_1, m_2, \ldots, m_s\rangle = \frac{1}{\sqrt{s!}} \det \begin{vmatrix} |m_1(1)\rangle & |m_2(1)\rangle & \cdots & |m_s(1)\rangle \\ \vdots & \vdots & \cdots & \vdots \\ |m_1(s)\rangle & |m_2(s)\rangle & \cdots & |m_s(s)\rangle \end{vmatrix} \tag{2.5}$$

where $m_1 > m_2 > \ldots > m_s$. The functions $|m_k(l)\rangle$ are eigenfunctions of the operator $R_z(\omega)$ of the rotation through an angle ω about the z axis and correspond to the eigenvalues $\exp(i\omega m_k)$. So we have

$$R_z(\omega)|m_1, m_2, \ldots, m_s\rangle = \exp\left(i\omega \sum_{k=1}^{s} m_k\right)|m_1, m_2, \ldots, m_s\rangle. \tag{2.6}$$

Using (2.6) one can easily find that the character of this rotation in the representation $\gamma^{[1^s]}$ is

$$\chi_{\gamma^{[1^s]}}(\omega) = \sum_{m_1>m_2>\ldots>m_s} z^{\sum_{k=1}^{s} m_k}. \tag{2.7}$$

Inserting this result first into (2.3) and then into (2.2), we obtain as an integrand the following sum:

$$S = \sum_{s=0}^{d} \sum_{m_1>m_2>\ldots>m_s} \lambda^s z^{\sum_{k=1}^{s} m_k}.$$

It can be easily seen that

$$\begin{aligned} S &= \prod_{m=-j}^{j} (1+\lambda z^m) \\ &= 1 + \sum_{m_1=-j}^{j} \lambda z^{m_1} + \sum_{m_1>m_2} \lambda^2 z^{m_1} z^{m_2} + \ldots + \lambda^d z^j z^{j-1} \ldots z^{-j}. \end{aligned} \tag{2.8}$$

Now we can pass over to the integration of this result over the classes of the group SO(3) (see, for example, Asherova *et al* 1988) and obtain the following expression of the fermion Molien functions:

$$\varphi(J, j; \lambda) = \frac{\mathrm{i}}{\pi} \int \frac{\mathrm{d}z(1-z)^2}{4z^2} \chi_J^*(z) \prod_{m=-j}^{j} (1+\lambda z^m). \tag{2.9}$$

The integration is carried out over the unit circle in a complex plane. Therefore, assuming that $\lambda < 1$, one can evaluate this integral by residues.

In the special case of the fermion MF for the invariants (i.e. in the case $J=0$) we have

$$\varphi(0, j; \lambda) = -\tfrac{1}{2} \mathrm{Res}|_{z=0} \frac{(1-z)^2}{z^2} \prod_{m=-j}^{j} (1+\lambda z^m). \tag{2.10}$$

For example in the case $j=1$, we obtain

$$\varphi(0, 1; \lambda) = 1+\lambda^3.$$

This means that there exist only the zero- and third-degree invariants. This result has a clear physical interpretation. The multiplicity $n(0, j, s)$ gives the number of antisymmetrized states with a total angular momentum $J=0$, which can be found in the shell model configuration j^s $(0 \leqslant s \leqslant 2j+1)$. It is well known that for p particles $(j=1)$ the antisymmetrical states with total angular momentum $J=0$ may exist only in configurations P^0 and P^3, which is in agreement with (2.10). By analogy, for the case $j=3$ we obtain

$$\varphi(0, 3; \lambda) = 1+\lambda^3+\lambda^4+\lambda^7.$$

Let us now consider the case of half-integer j. The integration over ω is carried out from 0 to 4π and it is convenient to substitute $z \to \sqrt{z}$. As a result, for the fermion MF we obtain

$$\varphi(J, j; \lambda) = \frac{\mathrm{i}}{4\pi} \int \frac{\mathrm{d}z(1-z^2)^2}{z^3} \chi_J^*(z^2) \prod_{m=-j}^{j} (1+\lambda z^{2m}). \tag{2.11}$$

In the special case of the invariants (i.e. $J=0$) the MF has the form

$$\varphi(0, j; \lambda) = -\tfrac{1}{2} \mathrm{Res}|_{z=0} \frac{(1-z^2)^2}{z^3} \prod_{m=-j}^{j} (1+\lambda z^{2m}).$$

Using this function, for example, in the case $j=\frac{7}{2}$ we obtain

$$\varphi(0,\tfrac{7}{2};\lambda)=1+\lambda^2+\lambda^4+\lambda^6+\lambda^8$$

which means that in this case the O(3) scalars exist only in the configurations $(\frac{7}{2})^2$, $(\frac{7}{2})^4$, $(\frac{7}{2})^6$ and $(\frac{7}{2})^8$.

In the above-considered cases one can also easily obtain expressions for the MF for the covariants (i.e. for the tensors of rank J). However, we prefer first to generalize the FM (2.9) and to represent it in the form which contains information about all tensors of rank J which can exist in the arbitrary antisymmetrized Kronecker products $j^{[1^s]}$, $s=0,1,\ldots,d$. For this reason we can introduce the generating functions of two parameters

$$\begin{aligned}\mathscr{F}(j;\lambda,t)&=\sum_{J=0}^{\infty}\varphi(J,j;\lambda)t^J\\&=\sum_{J=0}^{\infty}\sum_{s=0}^{d}n(J,j;s)\lambda^s t^J.\end{aligned}\tag{2.12}$$

The coefficient before the term $\lambda^s t^J$ in the expansion of these functions in powers of the parameters λ and t shows the number of tensors of rank J which can be found in the antisymmetrized sth Kronecker product $j^{[1^s]}$ of the representation D^j of SU(2). Taking into account that

$$\sum_{J=0}^{\infty}t^J\chi_J^*(z)=\frac{1}{z-1}\left(\frac{z}{1-zt}-\frac{z}{z-t}\right)=\frac{(1+t)z}{(z-t)(1-zt)}$$

it is obvious that the function $\mathscr{F}(j;\lambda,t)$ can also be calculated by the method of residues. So for integer values of j we have

$$\mathscr{F}(j;\lambda,t)=-\tfrac{1}{2}\mathrm{Res}\frac{(1-z)^2}{z(z-t)(1-zt)}\prod_{m=-j}^{j}(1+\lambda z^m)\tag{2.13}$$

and for half-integer values

$$\mathscr{F}(j;\lambda,t)=-\tfrac{1}{2}\mathrm{Res}\frac{(1-z^2)^2}{z^2(z-t)(1-zt)}\prod_{m=-j}^{j}(1+\lambda z^{2m}).\tag{2.14}$$

In the last case the series expansion of the GF

$$\bar{\mathscr{F}}(j;\lambda,t)=\sum n(J,j,s)\lambda^s t^k\qquad k=2J\tag{2.15}$$

differs from (2.12) by substituting $k=2J$ instead of J. Application of the formulae (2.13) and (2.14) to the particular cases $j=1$ and $j=\frac{3}{2}$ gives the following results

$$\mathscr{F}(1;\lambda,t)=1+\lambda t+\lambda^2 t+\lambda^3$$

$$\bar{\mathscr{F}}(\tfrac{3}{2};\lambda,t)=1+\lambda t^3+\lambda^2+\lambda^2t^4+\lambda^3t^3+\lambda^4.$$

These generating functions can by used for the classification of the antisymmetric states belonging to an arbitrary configuration j^n. Of course, the same results for many particular values of $j=\frac{1}{2}$, 1, $\frac{3}{2}$, etc, can be found in the tables, or can be obtained by means of other methods (for instance, by means of the characters of the antisymmetrized sth Kronecker products of IR (Ljubarsky 1957), or using combinatorial methods (see Buther 1967, Sunko and Syrtan 1985)). The significance of the generating functions, however, is that they contain the whole information of the tables in a compact form.

So far we have considered only the case of the group $G = SO(3)$ (SU(2)). This analysis, however, allows one to suggest that the two-parameter fermion MF for arbitrary group G can be written down in terms of the supertrace sTr and superdeterminant sdet that will be considered below

$$\varphi(\Gamma, \gamma; \lambda) = \frac{1}{V_G} \int \mathrm{d}g \frac{(\mathrm{sTr}\, D^{\Gamma}(g))^*}{\mathrm{sdet}|I - \lambda D^{\gamma}(g)|}.$$

Since the two-parameter boson MF has an evident form

$$\Phi(\Gamma, \gamma; \lambda) = \frac{1}{V_G} \int \mathrm{d}g \frac{(\mathrm{Tr}\, D^{\Gamma}(g))^*}{\det|I - \lambda D^{\gamma}(g)|}$$

we see that the transition from boson MF to fermion MF can be done through the usual Tr and det by supertraces and superdeterminants.

3. Molien functions for the SU(2) invariants and covariants in the enveloping UOSp($\frac{1}{2}$) algebra

Now, analysing the example of SU(2) invariants and covariants in the OSp($\frac{1}{2}$) superalgebra, let us consider the generalization of MF to the case of supersymmetry. As is well known (Berezin and Tolstoy 1981), the basic elements of the OSp($\frac{1}{2}$) superalgebra are the operators J_1, J_2, J_3 (the even generators of subalgebra SU(2)) and the operators R_+, r_- (the odd generators which form a tensor of rank $\frac{1}{2}$ according to SU(2)).

We can construct the generating function for the numbers of SU(2) invariants and covariants by the same formula (2.3), in which, however, $\chi_{\gamma^{[s]}}(d)$ is replaced by the character of the rotation g for the sth supersymmetric Kronecker product $\gamma^{\{s\}}$ of the representation $\gamma = D^1 \oplus D^{1/2}$ of the group SU(2). Here $\{s\}$ is a Young superdiagram containing s squares in one row. In the pure-boson case we have $\{s\} = [s]$, i.e. the supersymmetric Kronecker product coincides with the ordinary symmetrized Kronecker product, and in the fermion case the sth supersymmetric Kronecker product coincides with the corresponding antisymmetrized sth Kronecker product. In the 'mixed' boson-fermion case (i.e. in the space of the graduated representation γ) the space of the supersymmetric Kronecker product of the representation under consideration may be constructed in the following way.

Let us have three boson creation operators with spin 1, b_m^+, $m = 0, \pm 1$ and two fermion creation operators with spin $\frac{1}{2}$, a_μ^+, $\mu = \pm\frac{1}{2}$ and let us consider the set of all states of the type

$$|n_1, n_2, n_3, n_4, n_5\rangle = (b_1^+)^{n_1}(b_0^+)^{n_2}(b_{-1}^+)^{n_3}(a_{1/2}^+)^{n_4}(a_{-1/2}^+)^{n_5}|0\rangle \tag{3.1}$$

where $|0\rangle$ is the boson-fermion vacuum. It is obvious that the occupation numbers n_1, n_2, n_3 can be arbitrary non-negative integers, while the numbers n_4 and n_5 can take only the values 0 or 1. The space of the states (3.1) with $n_1 + n_2 + n_3 + n_4 + n_5 = s$ is isomorphic to the sth supersymmetric Kronecker product of the representation $\gamma = D^1 \oplus D^{1/2}$. On the other hand, the operator of the rotation can be represented in the form

$$R_3(\omega) = \exp(\mathrm{i}\omega J_3)$$

where

$$J_3 = \sum_m m b_m^+ b_m + \sum_\mu \mu a_\mu^+ a_\mu$$

and it is evident that the vectors (3.1) are eigenvectors of this operator and correspond to the eigenvalues

$$\exp[\mathrm{i}\omega(n_1-n_3+\tfrac{1}{2}n_4-\tfrac{1}{2}n_s)]=z^{2n_1-2n_3+n_4-n_5} \tag{3.2}$$

where $z=\exp(\mathrm{i}\omega/z)$. Therefore the character of the supersymmetric sth Kronecker product of the representation is equal to

$$\chi_{\gamma^{(s)}}(\omega)=\sum_{n_1,\ldots,n_5} z^{2n_1-2n_3+n_4-n_5} \tag{3.3}$$

where the summation is carried out over all n_i satisfying the condition

$$n_4,\, n_5=0 \text{ or } 1 \qquad \text{and} \qquad n_1+n_2+n_3+n_4+n_5=s.$$

Inserting this result into (2.3), and after that into (2.2), it is easy to find that the supersymmetric MF has the form

$$F(J,\gamma;\lambda)=\frac{1}{V_G}\int \mathrm{d}g\,\chi_J^*(g)\sum_s \lambda^s \sum_{n_1,\ldots,n_5} z^{2n_1-2n_3+n_4-n_5}. \tag{3.4}$$

Taking into acccount that $s=n_1+n_2+\ldots+n_5$, the sum in (3.4) can be rewritten in the form

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{1}\sum_{n_5=0}^{1}(\lambda z^2)^{n_1}\lambda^{n_2}(\lambda z^{-2})^{n_3}(\lambda z)^{n_4}(\lambda z^{-1})^{n_5}$$
$$=\frac{(1+\lambda z)(1+\lambda z^{-1})}{(1-\lambda z^2)(1-\lambda)(1-\lambda z^{-2})}.$$

In such a way we obtain

$$F(J,\gamma;\lambda)=\frac{1}{V_G}\int \mathrm{d}g\frac{(1+\lambda z)(1+\lambda z^{-1})}{(1-\lambda z^2)(1-\lambda)(1-\lambda z^{-2})}. \tag{3.5}$$

Now the MF (3.4) can be calculated again by residues.

In the special case of MF for the SU(2) invariants in the enveloping algebra UOSp($\frac{1}{2}$) we obtain

$$F(0,1\oplus\tfrac{1}{2};\lambda)=-\frac{1}{2(1-\lambda)}\operatorname*{Res}_{(z=0,\pm\sqrt{\lambda})}\frac{(1-z^2)^2(1+\lambda z)(z+\lambda)}{z^2(1-\lambda z^2)(z^2-\lambda)}. \tag{3.6}$$

In this case, however, as in all the other cases when the representation γ is reducible, it is convenient to use the generating functions not with one, but with two parameters—parameter λ_1 which gives the degree of the boson operator b^+ (or the degree of the even part of the representation, D^1) and parameter λ_2 which gives the degree of the fermion operator a^+ (or the degree of the odd part of the representation, $D^{1/2}$). This allows us to easily separate the even and the odd parts in space $\gamma^{\{s\}}$ and to determine which of these parts belongs to a given invariant or covariant. The two-parameter form of the formula (3.5) is

$$F(0,1\otimes\tfrac{1}{2};\lambda_1,\lambda_2)$$
$$=\frac{1}{2(1-\lambda_1)}\operatorname*{Res}_{(z=0,\pm\sqrt{\lambda_1})}\frac{(1-z^2)^2(1+\lambda_2 z)(z+\lambda_2)}{z^2(1-\lambda_1 z^2)(z^2-\lambda_1^2)}=\frac{1+\lambda_2^2}{1-\lambda_2^2}. \tag{3.7}$$

This means that in the enveloping algebra of UOSp($\frac{1}{2}$) there exists one basic invariant J^2 that can be used in arbitrary power and one auxiliary invariant $B=R_+R_-+R_-R_+$

which can appear only with exponent 0 or 1, so that the general SU(2) invariant H can be represented in the form

$$H=\sum_{n=0}^{\infty}(a_n+b_nB)J^{2n}. \tag{3.8}$$

By analogy with (2.14) one can construct GF for all covariants of $\mathrm{UOSp}(\frac{1}{2})$

$$\mathscr{F}(1\oplus\tfrac{1}{2};\lambda_1,\lambda_2,t)=\frac{1+\lambda_2t+\lambda_1\lambda_2t+\lambda_2^2}{(1-\lambda_2t^2)(1-\lambda_1^2)}. \tag{3.9}$$

Obviously this expression can also solve the problem of the reduction of the irreducible representation of the supergroup $\mathrm{U}(\frac{3}{2})$ with Young superdiagram $\{s\}$ when this group is restricted to the subgroups $\mathrm{O_B}(3)\times\mathrm{SU_F}(2)\rightarrow\mathrm{SU}_J(2)$.

It should be mentioned that in (3.9), and in (2.15) as well, the exponent of the parameter t^{2J} gives the doubled value of angular momentum J.

Now let us extend this particular result to the case when the general supergroup G′ contains, as a subgroup, the ordinary group G, $\mathrm{G}'\supset\mathrm{G}$. In this case we want to find the MF giving the number of invariants (or covariants of rank J) with respect to G which can be found in the sth supersymmetric Kronecker product of the representation γ of the supergroup G′. For this purpose one can generalize the expression (3.5) writing the MF for the above-mentioned case in the form

$$F(\Gamma,\gamma;\lambda)=\frac{1}{V_{\mathrm{G}}}\int \mathrm{d}g\frac{(\mathrm{sTr}\,D^{\Gamma}(g))^*}{\mathrm{sdet}|I-\lambda D^{\gamma}(g)|}. \tag{3.10}$$

Here, as in Berezin (1983), we use the following definition for the superdeterminant:

$$\mathrm{sdet}\begin{vmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{vmatrix}=\det|A_{11}-A_{12}A_{22}^{-1}A_{21}|\det A_{22}^{-1}. \tag{3.11}$$

In (3.11) the submatrix A_{11} corresponds to the space of the even states, and A_{22} corresponds to the space of the odd states. By definition, the supertrace of the matrix is

$$\mathrm{sTr}\begin{vmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{vmatrix}=\mathrm{Tr}\,A_{11}-\mathrm{Tr}\,A_{22} \tag{3.12}$$

which means that if the irreducible representation belongs to the even part of space $\gamma^{\{s\}}$ then $\mathrm{sTr}\,D^{\Gamma}(g)=\chi_{\Gamma}(g)$ and if D^{Γ} belongs to the odd part of space $\gamma^{\{s\}}$ then $\mathrm{sTr}\,D^{\Gamma}(g)=-\chi_{\Gamma}(g)$. As a matter of fact, if we take the matrix $D^{\gamma}(g)$ in diagonal form, then in the case $\gamma=D^1\oplus D^{1/2}$ the expression (3.10) gives

$$F(J,1\oplus\tfrac{1}{2};\lambda)=\frac{1}{V_{\mathrm{G}}}\int \mathrm{d}g(-1)^{2J}\chi_J^*(g)\frac{(1-\lambda z)(1-\lambda z^{-1})}{(1-\lambda z^2)(1-\lambda)(1-\lambda z^{-2})} \tag{3.13}$$

which is in agreement with (3.5). This can be easily seen taking into account that for the half-integer J on the RHS of (3.13) only the odd degrees of λ survive, and the sign factor $(-1)^{2J}=-1$ may be compensated if we substitute $\lambda\rightarrow-\lambda$, after which (3.13) coincides with (3.5).

As has been mentioned in (2.16), the formula (3.13) is valid also for the pure-fermion case. As a matter of fact, in this case

$$\mathrm{sdet}(1-\lambda D^j(g))=\left(\prod_{m=-j}^{j}(1-\lambda z^m)\right)^{-1} \qquad \mathrm{sTr}\,D^J(g)=\pm\chi_J(g)$$

where the sign in the expression for the supertrace is plus if the IR D^J appears in the even sth Kronecker product, and minus if s is odd.

Inserting these expressions in (3.10) and taking into account that the even degrees of λ appear with a plus sign, while the odd degrees of λ appear with a minus sign (which is equivalent to the substitution $\lambda \to -\lambda$), one can easily see that

$$\begin{aligned}\varphi(J,j;\lambda) &= \frac{\mathrm{i}}{\pi}\int \frac{\mathrm{d}z(1-z)^2}{4z^2}(\pm\chi_J^*(z)) \prod_{m=-j}^{j}(1-\lambda z^m) \\ &= \frac{\mathrm{i}}{\pi}\int \mathrm{d}z\frac{(1-z)^2}{4z^2}\chi_J^*(z) \prod_{m=-j}^{j}(1+\lambda z^m).\end{aligned}$$

The formula (3.10) turned out to be a sufficiently reasonable generalization of the boson MF (2.16) to the fermion case, as well as to the case of the graded space $\gamma = 1 \oplus \frac{1}{2}$. This is due to the fact that in these cases one of the following alternatives is possible: (i) the representation D^Γ is related to the even or to the odd parts of the space $\gamma^{\{s\}}$ (which corresponds to the integer or half-integer values of J); (ii) the space $\gamma^{\{s\}}$ in itself is entirely even or odd (as in the fermion case). In the more general case the formula (3.10) will give a function $F(\Gamma, \gamma; \lambda)$ whose expansion in powers of λ will have a meaning different from the ordinary MF. In this case we shall have

$$F(\Gamma, \gamma; \lambda) = \sum_s \nu(\Gamma, \gamma, s)\lambda^s$$

where

$$\nu(\Gamma, \gamma, s) = n_{\text{even}}(\Gamma, \gamma, s) - n_{\text{odd}}(\Gamma, \gamma, s).$$

Here $n_{\text{even(odd)}}(\Gamma, \gamma, s)$ are the multiplicities of IR Γ in the even and odd parts of the supersymmetric Kronecker product $\gamma^{\{s\}}$. So we lose part of the information in the spectrum of $\gamma^{\{s\}}$ when G′ is restricted to the subgroup G. To avoid this it is necessary to decompose the representation $\gamma = \gamma_e \oplus \gamma_o$ into even and odd parts and to introduce, by analogy with (3.7), two parameters λ_1 and λ_2 which control the exponents of the submodules γ_e and γ_o. The terms with the even exponents λ_1 belong to the even part of $\gamma^{\{s\}}$, and the odd exponents λ_2 are related to the odd part of $\gamma^{\{s\}}$. In this way it is possible to independently find $n_{\text{even}}(\Gamma, \gamma, s)$ and $n_{\text{odd}}(\Gamma, \gamma, s)$ and also the total multiplicity of the IR Γ in $\gamma^{\{s\}}$, i.e.

$$n(\Gamma, \gamma, s) = n_{\text{even}}(\Gamma, \gamma, s) + n_{\text{odd}}(\Gamma, \gamma, s) \tag{3.14}$$

as has been demonstrated in (3.9).

So far we have considered the GF for the restriction of some group or supergroup G′ to the ordinary subgroup G. In this way it has been shown that many aspects of the programme for the calculation of Molien functions, which was implemented by the Canadian theoreticians (Patera and Sharp 1979) for the ordinary groups $G' \supset G$, can be applied to the case when G′ is a supergroup as well. This makes it possible to solve, in an elegant way, many classification problems in group and supergroup representation theory. The problem of the construction of MF in the case when both G′ and G are supergroups is still open, because the characters of the representations of supergroups differ in properties from those of ordinary groups and expression (3.10) cannot be considered as a superanalogue of MF without additional considerations. We hope to come back to this question in the future and to consider in a forthcoming paper how generalized Molien functions might be used in problems such as the classification of super Lie algebras.

Acknowledgments

One of the authors (YuFS) expresses his gratitude to the INRNE for hospitality during his visit to the institute, where the major part of this paper was prepared. The authors also thank Dr G Kyrchev and Dr V N Tolstoy for valuable discussions.

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